Pathway Model, Superstatistics, Tsallis Statistics, and a Generalized Measure of Entropy

A.M. Mathai
Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Street West, Montreal, Canada H3A 2K6

H.J. Haubold
Office for Outer Space Affairs, United Nations, Vienna International Centre, P.O. Box 500, A-1400 Vienna, Austria

Abstract. The pathway model of Mathai (2005) is shown to be inferable from the maximization of a certain generalized entropy measure. This entropy is a variant of the generalized entropy of order α, considered in Mathai and Rathie (1975), and it is also associated with Shannon, Boltzmann-Gibbs, Rényi, Tsallis, and Havrda-Charvát entropies. The generalized entropy measure introduced here is also shown to have interesting statistical properties and it can be given probabilistic interpretations in terms of *inaccuracy measure*, *expected value*, and *information content* in a scheme. Particular cases of the pathway model are shown to be Tsallis statistics (Tsallis, 1988) and superstatistics introduced by Beck and Cohen (2003). The pathway model's connection to fractional calculus is illustrated by considering a fractional reaction equation.

1 Introduction

The fundamental problem pursued in equilibrium statistical mechanics is that given a large number of physical species, such as atoms, one wishes to know how they distribute according to some common property, e.g. velocity or energy (Ebeling and Sokolov, 2005). A simple mathematical model to understand the problem is at the center of statistics and probability theory. In order to deal with applications to physical situations of interest one takes into consideration the fundamental hypothesis of equal a priori probabilities for regions in phase space of an isolated system. This hypothesis is based on our insufficient knowledge for a specification of the precise state of the physical system under consideration. This hypothesis allows us

to assign systems to states that agree equally well with our knowledge of the actual condition of the system. This leads to the Boltzmann-Gibbs entropy, or Boltzmann principle as Einstein called it, $S = k\ lnW$, where W is the thermodynamic probability which is defined as the total number of equally probable microstates corresponding to the given macrostate. The Boltzmann constant is denoted by k. The Boltzmann-Gibbs entropy is relevant for situations such that all possible states of the system are considered equally probable. If we consider such a system in contact with a thermostat then we obtain the usual Maxwell-Boltzmann distribution for the possible states by maximizing the Boltzmann-Gibbs entropy S with the normalization and energy constraints. However, in nature many systems show distributions which differ from the Maxwell-Boltzmann distribution. These are usually systems with strong autocorrelations preventing the convergence to the Maxwell-Boltzmann distribution in the sense of the central-limit theorem. Well known examples in physics are: self gravitating systems, charged plasmas, Brownian particles in the presence of driving forces, and, more generally, non-equilibrium states of physical systems (Abe and Okamoto, 2001; Gell-Mann and Tsallis, 2004). Then it is natural to ask the question of whether non-Maxwell-Boltzmannian distributions can also be obtained from a corresponding maximum entropy principle, considering a generalized form for the entropy. For this purpose, different forms were proposed, as for instance the Tsallis entropy $S_q = \frac{W^{1-q}-1}{1-q}$, where q is the entropic index, that is considered the basis for a generalization of Boltzmann-Gibbs statistical mechanics (Abe and Okamoto, 2001; Gell-Mann and Tsallis, 2004). In the present paper we are investigating the link between entropic functionals and the corresponding families of distributions in Mathai's pathway model. We come to the conclusion that this link is also important to physically analyze fractional reaction equations in terms of probability theory.

The structure of the paper is the following: In Section 2 we introduce basic notions of Mathai's pathway model in terms of parametric families of distributions. In Section 3 we introduce a generalized entropic measure and investigate its characteristics and establish the link to parametric families of distributions in Mathai's pathway model, including Tsallis' distribution. In Section 4 we establish the link between a fractional reaction equation, its reaction coefficient considered a random variable, and Tsallis statistics and superstatistics.

2 Preliminaries for Mathai's pathway model

For practical purposes of analysing data of physical experiments and in building up models in statistical physics, we frequently select a member from a parametric family of distributions. It is often found that fitting experimental data needs a model with a thicker or thinner tail than the ones available from the parametric family, or a situation of right tail cut off (Honerkamp, 1994). The experimental data reveal that the underlying distribution is in between two parametric families of distributions. This observation either apeals to the form of the entropic functional or to the representation by a distribution function. In order to create a pathway from one functional form to another a pathway parameter is introduced and a pathway model is created in Mathai (2005). This model enables one to proceed from a generalized type-1 beta model to a generalized type-2 beta model to a generalized gamma model when the variable is restricted to be positive. More families are available when the variable is allowed to vary over the real line. Mathai (2005) deals mainly with rectangular matrix-variate distributions and the scalar case is a particular case there. For the real scalar case the pathway model is the following:

$$f(x) = cx^{\gamma-1}[1 - a(1 - \alpha)x^{\delta}]^{\frac{1}{1-\alpha}}, \tag{1}$$

$a > 0, \delta > 0, 1-a(1-\alpha)x^{\delta} > 0, \gamma > 0$ where c is the normalizing constant and α is the pathway parameter. For $\alpha < 1$ the model remains as a generalized type-1 beta model in the real case. For $a = 1, \gamma = 1, \delta = 1$ we have Tsallis statistics for $\alpha < 1$ (Tsallis, 1988, 2004). Other cases available are the regular type-1 beta density, Pareto density, power function, triangular and related models. Observe that (1) is a model with the right tail cut off. When $\alpha > 1$ we may write $1 - \alpha = -(\alpha - 1), \alpha > 1$ so that $f(x)$ assumes the form,

$$f(x) = cx^{\gamma-1}[1 + a(\alpha - 1)x^{\delta}]^{-\frac{1}{\alpha-1}}, \quad x > 0 \tag{2}$$

which is a generalized type-2 beta model for real x. Beck and Cohen's superstatistics belong to this case (2) (Beck and Cohen, 2003; Beck, 2006). For $\gamma = 1, a = 1, \delta = 1$ we have Tsallis statistics for $\alpha > 1$ from (2). Other standard distributions coming from this model are the regular type-2 beta, the F-distribution, Lévi models and related models. When $\alpha \to 1$ the forms in (1) and (2) reduce to

$$f(x) = cx^{\gamma-1}e^{-ax^{\delta}}, \quad x > 0. \tag{3}$$

This includes generalized gamma, gamma, exponential, chisquare, Weibull, Maxwell-Boltzmann, Rayleigh, and related models (Mathai, 1993a; Honerkamp, 1994). If x is replaced by $|x|$ in (1) then more families of distributions are covered in (1). The normalizing constant c for the three cases are available by putting $u = a(1-\alpha)x^{\delta}$ for $\alpha < 1$, $u = a(\alpha - 1)x^{\delta}$ for $\alpha > 1$, $u = ax^{\delta}$ for $\alpha \to 1$ and then integrating with the help of a type-1 beta integral, type-2 beta integral and gamma integral respectively. The value of c is the following:

$$
\begin{aligned}
c &= \frac{\delta[a(1-\alpha)]^{\frac{\gamma}{\delta}}\Gamma\left(\frac{\gamma}{\delta}+\frac{1}{1-\alpha}+1\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{1}{1-\alpha}+1\right)}, \quad \text{for } \alpha < 1 \\
&= \frac{\delta[a(\alpha-1)]^{\frac{\gamma}{\delta}}\Gamma\left(\frac{1}{\alpha-1}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{1}{\alpha-1}-\frac{\gamma}{\delta}\right)}, \quad \text{for } \frac{1}{\alpha-1}-\frac{\gamma}{\delta} > 0, \quad \alpha > 1 \\
&= \frac{\delta\, a^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)}, \quad \text{for } \alpha \to 1. \qquad (4)
\end{aligned}
$$

Observe that in (2) and (3), $\frac{1}{x}$ also belongs to the same family of densities and hence in (2) and (3) one could have also taken $x^{-\delta}$ with $\delta > 0$.

3 Pathway model from a generalized entropy measure

We introduce a generalized entropy measure here. This is a generalization of Shannon entropy and it is also a variant of the generalized entropy of order α in Mathai and Rathie (1975, 1976). Let us take the discrete case first. Consider a multinomial population $P = (p_1, ..., p_k), p_i \geq 0,\ i = 1, \ldots, k,\ p_1 + \ldots + p_k = 1$. Define the function

$$
\begin{aligned}
M_{k,\alpha}(P) &= \frac{\sum_{i=1}^{k} p_i^{2-\alpha} - 1}{\alpha - 1}, \quad \alpha \neq 1, \quad -\infty < \alpha < 2 \qquad (5) \\
\lim_{\alpha \to 1} M_{k,\alpha}(P) &= -\sum_{i=1}^{k} p_i \ln p_i = S_k(P) \qquad (6)
\end{aligned}
$$

by using L'Hospital's rule. In this notation $0 \ln 0$ is taken as zero when any $p_i = 0$. Thus (5) is a generalization of Shannon entropy $S_k(P)$ as seen from

(6). Note that (5) is a variant of Havrda-Charvát entropy $H_{k,\alpha}(P)$ and Tsallis entropy $T_{k,\alpha}(P)$ where

$$H_{k,\alpha}(P) = \frac{\sum_{i=1}^{k} p_i^{\alpha} - 1}{2^{1-\alpha} - 1}, \quad \alpha \neq 1, \quad \alpha > 0 \tag{7}$$

and

$$T_{k,\alpha}(P) = \frac{\sum_{i=1}^{k} p_i^{\alpha} - 1}{1-\alpha}, \quad \alpha \neq 1, \quad \alpha > 0. \tag{8}$$

We will introduce another measure associated with (5) and parallel to Rényi entropy $R_{k,\alpha}$ in the following form:

$$M_{k,\alpha}^{*}(P) = \frac{\ln\left(\sum_{i=1}^{k} p_i^{2-\alpha}\right)}{\alpha - 1}, \quad \alpha \neq 1, -\infty < \alpha < 2. \tag{9}$$

Rényi entropy is given by

$$R_{k,\alpha}(P) = \frac{\ln\left(\sum_{i=1}^{k} p_i^{\alpha}\right)}{1-\alpha}, \quad \alpha \neq 1, \quad \alpha > 0. \tag{10}$$

It will be seen later that the form in (5) is amenable to power law, pathway model etc. First we look into some basic properties enjoyed by $M_{k,\alpha}(P)$.

3.1 Properties

(i) Non-negativity: $M_{k,\alpha}(P) \geq 0$ with equality only when one $p_i = 1$ and the rest zeros.

(ii) Expansibility or zero-indifferent: $M_{k+1,\alpha}(P, 0) = M_{k,\alpha}(P)$. If an impossible event is incorporated into the scheme, that is, $p_{k+1} = 0$ it will not change the value of the entropy measure.

(iii) Symmetry: $M_{k,\alpha}(P)$ is a symmetric function of $p_1, \ldots, p_k$. Arbitrary permutations of $p_1, \ldots, p_k$ will not alter the value of $M_{k,\alpha}(P)$.

(iv) Continuity: $M_{k,\alpha}(P)$ is a continuous function of $p_i > 0, \ i = 1, \ldots, k$.

(v) Monotonicity: $M_{k,\alpha}\left(\frac{1}{k}, \ldots, \frac{1}{k}\right)$ is a monotonic increasing function of k.

(vi) Inequality: $M_{k,\alpha}(p_1,\ldots,p_k) \le M_{k,\alpha}\left(\frac{1}{k},\ldots,\frac{1}{k}\right)$.

(vii) Branching principle or recursivity:

$$\begin{aligned} M_{k,\alpha}(p_1,\ldots,p_k) &= M_{k-1,\alpha}(p_1+p_2,p_3,\ldots,p_k) + (p_1+p_2)^{2-\alpha} \times \\ &\times \; M_{2,\alpha}\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\right). \end{aligned}$$

This property indicates what happens to the measure if two of the mutually exclusive and totally exhaustive events, defining the multinomial population, are combined.

(viii) Non-additivity: Consider independent multinomial populations $P = (p_1,\ldots,p_n)$ and $Q = (q_1,\ldots,q_m)$ such that $\sum_{i=1}^n \sum_{j=1}^m p_i q_j = 1$, $\sum_{i=1}^n p_i = 1$, $\sum_{j=1}^m q_j = 1$. Then the joint density is of the form $(p_1q_1,\ldots, p_1q_m,\ldots,p_nq_1,\ldots,p_nq_m)$. Let us denote the entropy measure in this joint distribution by $M_{nm,\alpha}(P,Q)$. Then

$$M_{nm,\alpha}(P,Q) = M_{n,\alpha}(P) + M_{m,\alpha}(Q) + (\alpha-1)M_{n,\alpha}(P)M_{m,\alpha}(Q).$$

The third term on the right makes the measure non-additive. But the measures $M^*_{k,\alpha}(P)$ and Rényi entropy $R_{k,\alpha}(P)$ as well as Shannon entropy S_k are all additive. This is due to the fact that when logarithm of a product is taken it leads to a sum.

(ix) Decomposibility: Consider a joint discrete distribution $p_{ij} \ge 0$, $\sum_{i=1}^n \sum_{j=1}^m p_{ij} = 1$. Consider the marginal distribution $P_j = \sum_{i=1}^n p_{ij} > 0$, $j = 1,\ldots,m$. Then we have

$$\begin{aligned} & M_{nm,\alpha}(p_{11},p_{12},\ldots,p_{1m},p_{21},\ldots,p_{2m},\ldots,p_{n1},\ldots,p_{nm}) \\ = \; & M_{m,\alpha}(P_1,\ldots,P_m) + \sum_{j=1}^m P_j^{2-\alpha} M_{n,\alpha}\left(\frac{p_{1j}}{P_j},\ldots,\frac{p_{nj}}{P_j}\right). \end{aligned}$$

Observe that when $p_{ij} = p_i q_j$ with $\sum_{i=1}^n p_i = 1$, $\sum_{j=1}^m q_j = 1$ then property (ix) reduces to property $(viii)$.

(x) Functional equation: Consider $M_{2,\alpha}(P) = M_{2,\alpha}(p,1-p)$. That is,

$$M_{2,\alpha}(p,1-p) = \frac{p^{2-\alpha} + (1-p)^{2-\alpha} - 1}{\alpha - 1}, \;\; \alpha \ne 1, \; \alpha < 2.$$

Let $f_\alpha(p) = M_{2,\alpha}(p, 1-p)$. Then $f_\alpha(p)$ satisfies the functional equation

$$f_\alpha(x) + (1-x)^{2-\alpha} f_\alpha\left(\frac{y}{1-x}\right) = f_\alpha(y) + (1-y)^{2-\alpha} f_\alpha\left(\frac{x}{1-y}\right) \quad (11)$$

for $x, y \in [0,1), x+y \in [0,1]$ where $[\cdot)$ means open on the right and closed on the left, with

$$f_\alpha(0) = f_\alpha(1) = 0 \quad (12)$$

and

$$f_\alpha\left(\frac{1}{2}\right) = \frac{2^{\alpha-1}-1}{\alpha-1}, \quad \alpha \neq 1. \quad (13)$$

3.2 Characterization

Several characterization theorems can be established for the measure of generalized entropy in (5). We will indicate one such characterization. Let us look at an arbitrary continuous function, call it $f_\alpha(x)$, satisfying the functional equation in (11) with the boundary conditions in (12) and (13). What is the functional form of $f_\alpha(x)$? We will solve for $f_\alpha(x)$ from (11). Put $y = 1-x$ in (11) and use the boundary condition (12) to obtain

$$f_\alpha(x) = f_\alpha(1-x).$$

Take two numbers p and q in the open interval $(0,1)$. Put $p = 1-x$ and $q = y/(1-x)$ in (11). Then we obtain

$$f_\alpha(p) + p^{2-\alpha} f_\alpha(q) = f_\alpha(pq) + (1-pq)^{2-\alpha} f_\alpha\left(\frac{1-p}{1-pq}\right).$$

Let

$$F(p,q) = f_\alpha(p) + [p^{2-\alpha} + (1-p)^{2-\alpha}] f_\alpha(q).$$

Then it is easily seen that $F(p,q) = F(q,p)$, which implies that

$$f_\alpha(p) + [p^{2-\alpha} + (1-p)^{2-\alpha}] f_\alpha(q) = f_\alpha(q) + [q^{2-\alpha} + (1-q)^{2-\alpha}] f_\alpha(p).$$

Put $q = 1/2$ in this equation and use the boundary condition (13) to see that

$$f_\alpha(p) = \frac{p^{2-\alpha} + (1-p)^{2-\alpha} - 1}{\alpha - 1}.$$

Now we can build up successively by using the branching property $(viii)$.
Theorem 2.1. *Let $f_k(p_1, ..., p_k)$ be an arbitrary function of $p_1, \ldots, p_k$, $p_i \geq 0$, $i = 1, \ldots, k$, $p_1 + \ldots + p_k = 1$ satisfying the properties of symmetry (property (iii) of section 3.1), continuity (property (iv) of section 3.1), branching principle (property (vii) of section 3.1) where $f_2(p, 1-p) = f_\alpha(p)$, satisfies the functional equation (11) with the boundary conditions in (12) and (13) then $f_k(p_1, \ldots p_k)$ is uniquely determined as $M_{k,\alpha}(P)$ of (5).*

3.3 Continuous analogue

The continuous analogue to the measure in (5) is the following:

$$\begin{aligned} M_\alpha(f) &= \frac{\int_{-\infty}^{\infty} [f(x)]^{2-\alpha} dx - 1}{\alpha - 1} \qquad (14) \\ &= \frac{\int_{-\infty}^{\infty} [f(x)]^{1-\alpha} f(x) dx - 1}{\alpha - 1} = \frac{E[f(x)]^{1-\alpha} - 1}{\alpha - 1}, \quad \alpha \neq 1, \quad \alpha < 2 \end{aligned}$$

where $E[\cdot]$ denotes the expected value of $[\cdot]$. Note that when $\alpha = 1$, $E[f(x)]^{1-\alpha} = E[f(x)]^0 = 1$.
When $\alpha < 0$ and decreases then $1 - \alpha > 1$ and increases. The measure of uncertainty decreases in the discrete case when $\alpha < 0$. Similarly when $\alpha > 0$, then $1 - \alpha < 1$ and decreases. In the discrete case the measure of uncertainty increases. Hence we may call $1 - \alpha$ as the *strength of information* in the distribution. Larger the value of $1 - \alpha$ the larger the information content and smaller the uncertainlty and vice versa.

3.4 Connection to Kerridge's "inaccuracy" measure

A connection to Kerridge's measure of "inaccuracy" can also be explored here. Kerridge (1961) defined

$$K(P, Q) = -\sum_{i=1}^{n} p_i \ln q_i, \qquad (15)$$

$P = (p_1, \ldots, p_n)$, $Q = (q_1, \ldots, q_n)$, $p_i \geq 0$, $q_i \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^n p_i = 1, \sum_{i=1}^n q_i = 1$, as a measure of "inaccuracy" in assigning the multinomial distribution Q for the true distribution P. Here Q may be relative frequencies coming from an experiment or observations and P may be the true underlying distribution. A generalization of (15) can be the following:

$$M_\alpha(P, Q) = \frac{\sum_{i=1}^n p_i q_i^{1-\alpha} - 1}{\alpha - 1}, \quad \alpha \neq 1, \tag{16}$$

so that when $\alpha \to 1$ then $M_\alpha(P, Q)$ goes to $K(P, Q)$ of (15). Observe that we can also look upon $M_\alpha(P, Q)$ as an expected value with respect to the true multinomial distribution P and we may write it as

$$M_\alpha(P, Q) = \frac{E[q^{1-\alpha}] - 1}{\alpha - 1}, \quad \alpha \neq 1, \tag{17}$$

q taking the values $q_1, \ldots, q_n$ with the corresponding probabilities $p_1, \ldots, p_n$. The continuous analogue of (17) is the following:

$$M_\alpha(f_1, f_2) = \frac{E[f_2^{1-\alpha}] - 1}{\alpha - 1} = \frac{\int_{-\infty}^{\infty} f_1(x)[f_2(x)]^{1-\alpha} dx - 1}{\alpha - 1}, \quad \alpha \neq 1 \tag{18}$$

where the expectation is taken with respect to the density $f_1(x)$. Instead of $[f_2(x)]^{1-\alpha}$ if we assign $[f_1(x)]^{1-\alpha}$ through a displacement or distortion or disturbance to $f_1(x)$ then a measure of "inaccuracy" in assigning $[f(x)]^{1-\alpha}$ for $f(x)$ is given by $M_\alpha(f)$ of (14). More "inaccuracy" means less "information content" and more "uncertainty" or more "entropy", and vice versa. Again $1 - \alpha$ corresponds to the strength of information content.

The generalized entropy $M_\alpha(f)$ is evaluated for some standard distributions and given in the appendix at the end of this paper.

3.5 Distributions with maximum generalized entropy

Among all densities, which one will give a maximum value for $M_\alpha(f)$? Consider all possible functions $f(x)$ such that $f(x) \geq 0$ for all x, $f(x) = 0$ outside (a, b), $a < b$, $f(a)$ is the same for all such $f(x)$, $f(b)$ is the same for all such f, $\int_a^b f(x) dx = 1$. Let $f(x)$ be a continuous function of x possessing continuous

derivatives with respect to x. Then for using calculus of variation techniques consider

$$U = [f(x)]^{2-\alpha} - \lambda \ \ f(x). \tag{19}$$

Note that for fixed $\alpha, \alpha \neq 1$, maximization of $\frac{\int_a^b [f(x)]^{2-\alpha} dx - 1}{\alpha - 1}, \ \ \alpha \neq 1, \alpha < 2$ is equivalent to maximizing $\int_a^b [f(x)]^{2-\alpha} dx$. If necessary, we may also take

$$M_\alpha(f) = \frac{\int_a^b [f(x)]^{2-\alpha} dx}{\alpha - 1} - \frac{\int_a^b f(x) dx}{\alpha - 1}, \ \ \alpha \neq 1, \ \ \alpha < 2$$

since $\int_a^b f(x)dx = 1$. This will produce only a change in the Lagrangian multiplier λ in U above. Hence without loss of generality the form of U is as given in (19). We are looking at all possible f for every given x and α. Hence the Euler equation becomes,

$$\begin{aligned} \frac{\partial U}{\partial f} = 0 \ \ &\Rightarrow \ \ (2-\alpha)[f(x)]^{1-\alpha} - \lambda = 0 \\ &\Rightarrow \ \ f(x) = \frac{\lambda}{2-\alpha}, \end{aligned}$$

free of x, $\alpha < 2, \ \ \alpha \neq 1$. Thus $f(x)$ in this case is a uniform density over $[a, b]$.

Let us consider the situation where $E[x^\delta]$ for some δ is a fixed quantity for all such f. Then we have to maximize

$$\frac{\int_a^b [f(x)]^{2-\alpha} dx}{\alpha - 1} - \frac{1}{\alpha - 1}$$

subject to the conditions $\int_a^b f(x)dx = 1$ and $\int_a^b x^\delta f(x) dx$ is a given quantity. Consider

$$U = [f(x)]^{2-\alpha} - \lambda_1 f(x) + \lambda_2 x^\delta f(x).$$

Then the Euler equation is the following:

$$\begin{aligned} \frac{\partial U}{\partial f} = 0 \ \ &\Rightarrow \ \ (2-\alpha)[f(x)]^{1-\alpha} - \lambda_1 + \lambda_2 x^\delta = 0 \\ &\Rightarrow \ \ [f(x)]^{1-\alpha} = \frac{\lambda_1}{2-\alpha}[1 - \frac{\lambda_2}{\lambda_1} x^\delta] \\ &\Rightarrow \ \ f(x) = c_1 [1 - c_2 x^\delta]^{\frac{1}{1-\alpha}} \end{aligned} \tag{20}$$

where c_1 and c_2 are constants and $c_1 > 0, 1 - c_2 x^\delta > 0$ since it is assumed that $f(x) \geq 0$ for all x. When $c_2 = \beta(1-\alpha),\ \ \beta > 0$, we have

$$f(x) = c_1[1 - \beta(1-\alpha)x^\delta]^{\frac{1}{1-\alpha}}. \tag{21}$$

Then for $\delta = 1$ we have the power law

$$\frac{\partial f}{\partial x} = -c_3 f^\alpha \tag{22}$$

where c_3 is a constant. The form in (21) for $\alpha < 1$ remains as a special case of a generalized type-1 beta model; for $\alpha > 1$ it is a special case of a generalized type-2 beta model and when $\alpha \to 1$ it is a special case of a generalized gamma model when the range (a, b) is such that $a = 0, b = \infty$. For $\delta = 1$, (21) gives Tsallis statistics (Tsallis, 1988, 2004).
Observe that the generalized entropy $M_\alpha(f)$ of (14) gives rise to the power law with exponent α, readily, as seen from (22). Also notice that by selecting λ_1 and λ_2 in (20) we can obtain functions of the following forms also:

$$(1 - \beta_1 x^\delta)^{-\gamma_1} \text{ and } (1 + \beta_2 x^\delta)^{\gamma_2},\ \ \beta_1, \beta_2, \gamma_1, \gamma_2 > 0.$$

Both these forms are ever increasing and cannot produce densities in $(0, \infty)$ unless the range of x with nonzero $f(x)$ is finite.
In section 3.4 we have given several interpretations for $1 - \alpha$. We can also derive the pathway model by maximizing $M_\alpha(f)$ over all non-negative integrable functions. Consider all possible $f(x)$ such that $f(x) \geq 0$ for all x, $\int_a^b f(x)dx < \infty$, $f(x)$ is zero outside (a, b), $f(a)$ is the same for all $f(x)$, and similarly $f(b)$ is also the same for all such functional f. Let $f(x)$ be a continuous function of x with continuous derivatives in (a, b). Let us maximize $\int_a^b [f(x)]^{2-\alpha} dx$ for fixed α and over all functional f, under the conditions that the following two moment-like expressions be fixed quantities:

$$\int_a^b x^{(\gamma-1)(1-\alpha)} f(x)dx = \text{given, and } \int_a^b x^{(\gamma-1)(1-\alpha)+\delta} f(x)dx = \text{given} \tag{23}$$

for fixed $\gamma > 0$ and $\delta > 0$. Consider

$$U = [f(x)]^{2-\alpha} - \lambda_1 x^{(\gamma-1)(1-\alpha)} f(x) + \lambda_2 x^{(\gamma-1)(1-\alpha)+\delta} f(x),\ \ \alpha < 2,\ \ \alpha \neq 1$$

where λ_1 and λ_2 are Lagrangian multipliers. Then the Euler equation is the following:

$$\frac{\partial U}{\partial f} = 0 \quad \Rightarrow \quad (2-\alpha)[f(x)]^{1-\alpha} - \lambda_1 x^{(\gamma-1)(1-\alpha)} + \lambda_2 x^{(\gamma-1)(1-\alpha)+\delta} = 0$$

$$\Rightarrow \quad [f(x)]^{1-\alpha} = \frac{\lambda_1}{(2-\alpha)} x^{(\gamma-1)(1-\alpha)}[1 - \frac{\lambda_2}{\lambda_1} x^\delta] \tag{24}$$

$$\Rightarrow \quad f(x) = c_1 x^{\gamma-1}[1-\beta(1-\alpha)x^\delta]^{\frac{1}{1-\alpha}} \tag{25}$$

where λ_1/λ_2 is written as $\beta(1-\alpha)$ with $\beta > 0$ such that $1-\beta(1-\alpha)x^\delta > 0$ since $f(x)$ is assumed to be non-negative. By using the conditions (23) we can determine c_1 and β. When the range of x for which $f(x)$ is nonzero is $(0,\infty)$ and when c_1 is a normalizing constant then (25) is the pathway model of Mathai (2005) in the scalar case where α is the pathway parameter. When $\gamma = 1, \delta = 1$ then (25) produces the power law. The form in (24) for various values of λ_1 and λ_2 can produce all the four forms

$$\alpha_1 x^{\gamma-1}[1-\beta_1(1-\alpha)x^\delta]^{-\frac{1}{1-\alpha}},\ \alpha_2 x^{\gamma-1}[1-\beta_2(1-\alpha)x^\delta]^{\frac{1}{1-\alpha}} \text{ for } \alpha < 1$$

and

$$\alpha_3 x^{\gamma-1}[1+\beta_3(\alpha-1)x^\delta]^{-\frac{1}{\alpha-1}},\ \alpha_4 x^{\gamma-1}[1+\beta_4(\alpha-1)x^\delta]^{\frac{1}{\alpha-1}} \text{ for } \alpha > 1$$

with $\alpha_i, \beta_i > 0, i = 1,2,3,4$. But out of these, the second and the third forms can produce densities in $(0,\infty)$. The first and fourth will not be converging. When $f(x)$ is a density in (25) what is the normalizing constant c_1? We need to consider three cases of $\alpha < 1, \alpha > 1$ and $\alpha \to 1$. This c_1 is already evaluated in section 2.

4 Reaction equation, superstatistics, and fractional calculus

4.1 Reaction equation and fractional calculus

A reaction problem was examined by Haubold and Mathai (1995, 2000) where the number density of the reacting particles is a function of time t. For the i-th particle let the number density be denoted by $N_i(t)$ with $N_i(t=0) = N_0^{(i)}$.

If the production rate is proportional to the number density then we have

$$\frac{d}{dt}N_i(t) = a_i N_i(t)$$

where a_i is the reaction coefficient. The reaction coefficient itself can be considered to be a statistical quantity subject to accommodating a distribution function based either on Boltzmann-Gibbs or Tsallis statistics (Haubold and Mathai, 1998; Coraddu at al., 1999; Saxena, Mathai, and Haubold 2004b). If some of the particles produced are also destroyed and if the destruction rate is b_i for the i-th particle, and proportional to the number density, then

$$\frac{d}{dt}N_i(t) = b_i N_i(t).$$

If destruction dominates then the net residual effect is given by

$$\frac{d}{dt}N_i(t) = -c_i N_i(t),\ \ c_i = b_i - a_i > 0$$

and the solution is

$$N_i(t) = N_0^{(i)} e^{-c_i t},\ \ c_i > 0, t \geq 0.$$

But if a fractional integral, instead of the integer-order integral, is used then the residual reaction equation is given by

$$N_i(t) = N_0^{(i)} - c_i^{\nu}\ \ {}_0D_t^{-\nu} N_i(t),\ \ \nu > 0 \tag{26}$$

where the fractional integral operator is given by

$${}_aD_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t-u)^{\nu-1} f(u) du,\ \ \nu > 0, \tag{27}$$

with ${}_aD_t^0 f(t) = f(t)$. Note that in (26), c_i is replaced by c_i^{ν} for physical reason. If the production rate dominates over the destruction rate then in (26), c_i^{ν} is to be replaced by $-c_i^{\nu}$. For a discussion of general growth-decay models see Mathai (1993b). The Laplace transform of $N_i(t)$ coming from (26) is

$$L_{N_i(t)}(s) = \frac{N_0^{(i)}}{s\left[1 + \left(\frac{c_i}{s}\right)^{\nu}\right]} \tag{28}$$

and the solution can be written in terms of a Mittag-Leffler function:

$$N_i(t) = N_0^{(i)} \sum_{k=0}^{\infty} \frac{(-1)^k[(c_i t)^\nu]^k}{\Gamma(1+k\nu)} = N_0^{(i)} \;\; E_\nu(-c_i^\nu t^\nu) \tag{29}$$

where the generalized Mittag-Leffler function $E^{\gamma}_{\mu,\nu}(z)$ is defined by

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k \; z^k}{k!\Gamma(\beta+\alpha k)}, \;\; \beta > 0, \;\; \alpha > 0 \tag{30}$$

where $(\gamma)_k = \gamma(\gamma+1)...(\gamma+k-1), \;\; (\gamma)_0 = 1, \;\; \gamma \neq 0$ and

$$E^{1}_{\alpha,1}(z) = E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \;\; \alpha > 0.$$

Consider the situation of having a total of γ different particles, $\gamma = 1, 2, \ldots$, these particles reacting independently with the same rate $c_1 = c_2 = \ldots = c_\gamma = c$. For independent variables the Laplace transform of the sum is the product of Laplace transforms. Hence in this case $N(t) = N_1(t) + ... + N_\gamma(t)$ has the Laplace transform

$$L_{N(t)}(s) = \frac{N_0^{(1)} \ldots N_0^{(}\gamma)}{s^\gamma \left[1 + \left(\frac{c}{s}\right)^\nu\right]^\gamma} \tag{31}$$

which yields the solution

$$N(t) = N_0^{(1)} \ldots N_0^{(\gamma)} \;\; t^{\gamma-1} E^{\gamma}_{\nu,\gamma}(-t^\nu c^\nu). \tag{32}$$

The rate c may vary and c may be considered to have its own distribution. In this case $N(t)$ in (32) is to be taken as $N(t|c)$ or $N(t)$ at a given c. Superstatistics, as considered by Beck and Cohen (2003) and Beck (2006), will result when we use an appropriate prior density for c as developed in the following section.

4.2 Superstatistics and fractional calculus

Beck and Cohen developed the concept of superstatistics, see Beck and Cohen (2003) and Beck (2006). They showed that superstatistics considerations

are relevant in problems of Lagrangian and Eulerian turbulence, defect turbulence, atmospheric turbulence, cosmic ray statistics, solar flares, and solar wind statistics. From a statistical point of view, the procedure is equivalent to starting with a conditional distribution for every given value of a parameter β. Then β is assumed to have a prior known density. Then the unconditional distribution is obtained by integrating out over the density of β.

Reaction equations are considered in a series of papers by Haubold, Mathai, and Saxena (Haubold and Mathai, 1995, 2000; Saxena, Mathai, and Haubold, 2004a; Mathai, Saxena, and Haubold, 2005). For the sake of illustration we consider one such reaction equation which yields the number density of the following form, where γ and μ are arbitray, need not be integers. [Note that in (32) the resulting γ is a positive integer.]

$$\begin{aligned} N(t|c) &= N_0\, t^{\mu-1} E_{\nu,\mu}^{\gamma+1}(-c^{\nu}t^{\nu}), \quad \mu > 0,\ \gamma > 0,\ \nu > 0 \\ &= N_0\, t^{\mu-1} \sum_{k=0}^{\infty} \frac{(\gamma+1)_k(-1)^k(c^{\nu}t^{\nu})^k}{k!\ \Gamma(\mu+\nu k)}. \end{aligned} \tag{33}$$

Let us consider the situation where c in equation (26) is a random variable having a gamma type density:

$$g(c) = \frac{\omega^{\mu}}{\Gamma(\mu)} c^{\mu-1} e^{-\omega c},\ \omega > 0,\ 0 < c < \infty \tag{34}$$

where $\mu > 0, \omega, \mu$ are known and μ/ω is the mean value of c. The residual rate of change may have small probabilities of it being too large or too small and the maximum probability may be for a medium range of values for the residual rate of change c. This is a very reasonable asumption. Equation (33) is the situation where the residual rate of change is such that the production rate dominates so that we have the form $-c^{\nu}, \nu > 0, c > 0$. If the destruction rate dominates then the constant will be of the form $c^{\nu}, c > 0, \nu > 0$. Integrating out $N(t|c)$ over $g(c)$ we have

$$\int_{c=0}^{\infty} N(t|c)g(c)dc = N_0 \frac{\omega^{\mu}}{\Gamma(\mu)} t^{\mu-1} \sum_{k=0}^{\infty} \frac{(\gamma+1)_k(-1)^k(t^{\nu})^k}{k!\Gamma(\mu+\nu k)} \int_0^{\infty} c^{\mu-1+k\nu} e^{-c\omega} dc$$

where the integral over c yields $\Gamma(\mu+k\nu)\omega^{-(\mu+k\nu)}$. Hence the unconditional

number density, denoted by $N(t)$, is given by

$$\begin{aligned} N(t) &= \frac{N_0}{\Gamma(\mu)} t^{\mu-1} \sum_{k=0}^{\infty} \frac{(\gamma+1)_k (-1)^k}{k!} \left(\frac{t^\nu}{\omega^\nu}\right)^k \\ &= \frac{N_0}{\Gamma(\mu)} t^{\mu-1} \left(1+\frac{t^\nu}{\omega^\nu}\right)^{-(\gamma+1)} \text{ for } |\frac{t}{\omega}|<1. \end{aligned}$$

From the analytic continuation we see that the form remains the same for $\frac{t}{\omega}>1$ also. Hence

$$N(t) = \frac{N_0}{\Gamma(\mu)} t^{\mu-1} \left(1+\frac{t^\nu}{\omega^\nu}\right)^{-(\gamma+1)}, \; 0<t<\infty, \; \omega>0. \tag{35}$$

The continuation part may be seen by writing equation (33) with the help of the Mellin-Barnes representation in the form

$$N(t|c) = N_0 t^{\mu-1} \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\Gamma(1-s)\Gamma(s)}{\Gamma(1-s\nu)} (c^\nu t^\nu)^{-s} ds, \; i=\sqrt{-1}. \tag{36}$$

Then integrating over $g(c)$ of (34) we have

$$N(t) = N_0 t^{\mu-1} \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \Gamma(1-s)\Gamma(s) \left(\frac{t^\nu}{\omega^\nu}\right)^{-s} ds \tag{37}$$

where b is a real number, $0<b<1$. Evaluate the integral in (37) as the sum of the residues at the poles of $\Gamma(s)$, which are at $s=-k, k=0,1,...,$ to obtain the series form for $\frac{t}{\omega}<1$. Evaluate at the poles of $\Gamma(1-s)$ to obtain the analytic continuation for $\frac{t}{\omega}>1$. Both will lead to the same form in (35), which is a generalized type-2 beta form.

We may make the substitution $\gamma+1=\frac{1}{\alpha-1}, \alpha>1 \Rightarrow \gamma=\frac{\alpha-2}{\alpha-1}$ and $\omega^{-\nu}=b(\alpha-1), b>0$. Then we have the unconditional number density

$$N(t) = \frac{N_0}{\Gamma(\mu)} t^{\mu-1} [1+b(\alpha-1)t^\nu]^{-\frac{1}{\alpha-1}} \tag{38}$$

for $\alpha>1, t>0, b>0, \mu>0$. For $\mu=1, \nu=1, b=1$, equation (38) corresponds to Tsallis statistics for one case, namely, $\alpha=q>1$ (Tsallis 2004). For general values of μ and $\alpha>1$ such that $\frac{1}{\alpha-1}-\frac{\mu}{\nu}>0$ equation (38) corresponds to the pathway model of Mathai (2005) as well as the superstatistics

considered by Beck and Cohen (2003) and Beck (2006).
Now, consider an equation parallel to equation (17) in Mathai, Saxena, and Haubold (2005), namely,

$$N(t|c) - N_0 t^{\mu-1} E_{\nu,\mu}^{-\gamma}(c^\nu t^\nu) = c^\nu \ {}_0D_t^{-\nu} N(t) \tag{39}$$

for $\gamma > 0, c > 0, \nu > 0, \mu > 0$. Note that sets of parameters, $\gamma > 0, \nu > 0, \mu > 0$ exist so that the series form in $E_{\nu,\mu}^{-\gamma}(c^\nu t^\nu)$ is convergent and remains positive for all $t > 0$. This corresponds to the situation where the residual rate of change is positive so that the production dominates over destruction. We can expect $N(t|c)$ to increase to ∞. But the density of c in (34) will produce a dampening effect. Proceeding as before, we will obtain the final unconditinal $N(t)$ as follows:

$$N(t) = \frac{N_0}{\Gamma(\mu)} t^{\mu-1} \left[1 - \frac{t^\nu}{\omega^\nu}\right]^{\gamma-1} \quad \text{for } 0 < \frac{t}{\omega} < 1. \tag{40}$$

For convenience, we can write $\gamma - 1 = \frac{1}{1-\alpha}, \alpha < 1 \Rightarrow \gamma = \frac{2-\alpha}{1-\alpha}$ and $\omega^{-\nu} = b(1-\alpha), b > 0, \alpha < 1$ so that (40) becomes

$$N(t) = \frac{N_0}{\Gamma(\mu)} t^{\mu-1} [1 - b(1-\alpha)t^\nu]^{\frac{1}{1-\alpha}}, \tag{41}$$

for $\alpha < 1, b > 0, \nu > 0, \mu > 0, 0 < t^\nu < [b(1-\alpha)]^{-1}$. If $N(t)$ is to be made into a statistical density then we may normalize (38) with the help of a type-2 beta integral, and (41) with the help of a type-1 beta integral. When $\alpha \to 1$ both (38) and (41) will approach the generalized gamma form

$$N(t) = \frac{N_0}{\Gamma(\mu)} t^{\mu-1} e^{-bt^\nu}, \ b > 0, \mu > 0, \nu > 0, t > 0. \tag{42}$$

Equations (38), (41) and (42) are special scalar cases of the pathway model of Mathai (2005). For $\mu = 1, \nu = 1, b = 1$ Tsallis statistics is recovered from (38) and (41), with appropriate normalizations.
Remark 1. Superstatistics considerations can produce only the generalized type-2 beta form of (35) with the appropriate normalization constant because Beck and Cohen (2003) and Beck (2006) consider the situation of a conditional density belonging to the generalized gamma family where a scaling parameter is having a marginal distribution belonging to a generalized

gamma family. Then the unconditional density can only produce a form of the type

$$f(x) = c_1 x^{\alpha}(1+\beta_1 x^{\delta})^{-\gamma_1},\ c_1>0, \beta_1>0, \gamma_1>0, x>0 \tag{43}$$

which is a generalized type-2 beta form. A type-1 beta form as in (40) will not be available from such a procedure because the exponents will only sum up to give a form of the type in (43). A more general format of the Beck-Cohen type consideration, from a statistical point of view, is provided in the following section.

4.3 A more general density giving rise to generalized superstatistics

Consider a conditional density of the form

$$f_x(x|\beta) = c_2(x-x_o)^{\alpha-1}e^{-(\beta-\beta_o)^{\eta}(x-x_o)^{\delta}}, \tag{44}$$

for $\eta>0, \delta>0, \alpha>0, \beta>\beta_o, x>x_o$ where the normalizing constant c_2 can be seen to be the following:

$$c_2 = \frac{\delta\,(\beta-\beta_o)^{\frac{\alpha\eta}{\delta}}}{\Gamma\left(\frac{\alpha}{\delta}\right)}. \tag{45}$$

For various values of the parameters $\alpha, \eta, \delta, x_o, \beta_o$ we have the exponential, gamma, chisquare, Erlang, Helley, Helmert, Maxwell-Boltzmann, Rayleigh, and Weibull densities as special cases of (44) (Mathai, 1993a; Honerkamp, 1994). If $x-x_o$ is replaced by $|x-x_o|$ for $-\infty<x<\infty$ then many more densities such as double exponential, Laplace and Gaussian will be particular cases. Consider a prior density for the scaling parameter β in the following form:

$$g(\beta) = c_3(\beta-\beta_o)^{\gamma-1}e^{-\zeta(\beta-\beta_o)^{\eta}}, \tag{46}$$

for $\zeta>0, \eta>0, \gamma>0$ known, $\beta>\beta_o$ and $c_3 = \eta\zeta^{\frac{\gamma}{\eta}}/\Gamma\left(\frac{\gamma}{\eta}\right)$. This is a very general class of the type in (44). The only restriction imposed for convenience is that the exponents involving β are the same in (44) and (46). Then the unconditional density of x is given by

$$f_x(x) \;=\; \int_{\beta=\beta_o}^{\infty} f_x(x|\beta)g(\beta)d\beta$$

$$
\begin{aligned}
&= \frac{\eta\,\delta\,\zeta^{\frac{\gamma}{\eta}}(x-x_o)^{\alpha-1}}{\Gamma\left(\frac{\gamma}{\eta}\right)\Gamma\left(\frac{\alpha}{\delta}\right)}\int_{\beta=\beta_o}^{\infty}(\beta-\beta_o)^{\gamma+\frac{\alpha\eta}{\delta}-1}e^{-(\beta-\beta_o)^{\eta}[\zeta+(x-x_o)^{\delta}]}d\beta \\
&= \frac{\delta\,\Gamma\left(\frac{\gamma}{\eta}+\frac{\alpha}{\delta}\right)\,(x-x_o)^{\alpha-1}\left[1+\frac{(x-x_o)^{\delta}}{\zeta}\right]^{-\left(\frac{\gamma}{\eta}+\frac{\alpha}{\delta}\right)}}{\zeta^{\frac{\alpha}{\delta}}\Gamma\left(\frac{\gamma}{\eta}\right)\Gamma\left(\frac{\alpha}{\delta}\right)} \qquad (47)
\end{aligned}
$$

for $x > x_o, \zeta > 0, \gamma > 0, \eta > 0, \alpha > 0, \delta > 0$. Note that (47) can only produce a form of the type (43) or a generalized type-2 beta type. Type-1 beta form of (40) is not available from a procedure such as the steps in (44) to (47). This is not covered in Beck-Cohen's type superstatistics considerations.

Remark 2. Observe that if a real random variable x has a density belonging to the generalized gamma family then $\frac{1}{x}$ also belongs to a generalized gamma family. Hence in (44) and (46) we could have taken $(x-x_o)^{-\delta}$ with $\delta > 0$ and $(\beta-\beta_o)^{-\eta}$ with $\eta > 0$ in the exponents, individually or simultaneously. The procedure will remain the same and we will again end up with a generalized type-2 beta form corresponding to (47).

References

Abe, S. and Okamoto, Y. (Eds.)(2001). *Nonextensive Statistical Mechanics and Its Applications*. Springer-Verlag, Berlin and Heidelberg.

Beck, C. (2006). Stretched exponentials from superstatistics. *Physica*, **A 365**, 96-101.

Beck, C. and Cohen, E.G.D. (2003). Superstatistics. *Physica*, **A322**, 267-275.

Coraddu, M., Kaniadakis, G., Lavagno, A., Lissia, M., Mezzorani, G., and Quarati, P. (1999). Thermal distributions in stellar plasmas, nuclear reactions and solar neutrinos. *Braz. J. Phys.*, **29**, 153-168 [see also Lissia, M. and Quarati, P. (2005). Nuclear astrophysical plasmas: ion distribution functions and fusion rates. *Europhysics News*, **November/December**, 211-214].

Ebeling, W. and Sokolov, I.M. (2005). *Statistical Thermodynamics and Stochastic Theory of Nonequilibrium Systems*. World Scientific, Singapore.

Gell-Mann, M. and Tsallis, C. (Eds.)(2004). *Nonextensive Entropy: Interdisciplinary Applications*. Oxford University Press, New York.

Haubold, H.J. and Mathai, A.M. (1995). A heuristic remark on the periodic variation in the number of solar neutrinos detected on Earth. *Astrophysics and Space Science*, **228**, 113-134.

Haubold, H.J. and Mathai, A.M. (1998). An integral arising frequently in astronomy and physics. *SIAM Review*, **40**, 995-997.

Haubold, H.J. and Mathai, A.M. (2000). The fractional kinetic equation and thermonuclear functions. *Astrophysics and Space Science*, **273**, 53-63.

Honerkamp, J. (1994). *Stochastic Dynamical Systems: Concepts, Numerical Methods, Data Analysis*. VCH Publishers, New York.

Kerridge, D.F. (1961). Inaccuracy and inference. *Journal of the Royal Statistical Society, Series B*, **23**, 184-194.

Mathai, A.M. (1993a). *A Handbook of Generalized Special Functions for Statistical and Physical Sciences*, Clarendon Press, Oxford.

Mathai, A.M. (1993b). The residual effect of a growth-decay mechanism and the distributions of covariance structures. *The Canadian Journal of Statistics*, **21(3)**, 277-283.

Mathai, A.M. (2005). A pathway to matrix-variate gamma and normal densities. *Linear Algebra and Its Applications*, **396**, 317-328.

Mathai, A.M. and Rathie, P.N. (1975). *Basic Concepts in Information Theory and Statistics: Axiomatic Foundations and Applications*, Wiley Halsted, New York and Wiley Eastern, New Delhi.

Mathai, A.M. and Rathie, P.N. (1976). Recent contributions to axiomatic definitions of information and statistical measures through functional equations. In *Essays in Probability and Statistics* (eds. S. Ikeda et al.), Shinko Tsusho Co. Ltd., Tokyo, Japan, pp. 607-633.

Mathai, A.M., Saxena, R.K. and Haubold, H.J. (2005). A certain class of Laplace transforms with applications to reaction and reaction-diffusion equations. *arXiv. math.CA/0604472 v2*.

Saxena, R.K., Mathai, A.M., and Haubold, H.J. (2004a). On generalized fractional kinetic equations. *Physica*, **A 344**, 657-664.

Saxena, R.K., Mathai, A.M., and Haubold, H.J. (2004b). Astrophysical thermonuclear functions for Boltzmann-Gibbs statistics and Tsallis statistics. *Physica*, **A 344**, 649-656.

Tsallis, C. (1988). Possible generalization of Boltzmann-Gibbs statistics. *J. Stat. Phys.*, **52**, 479-487.

Tsallis, C. (2004). What should a statistical mechanics satisfy to reflect nature? *Physica D*, **193**, 3-34.

Appendix

A1. Evaluation of the generalized entropy for some standard distributions

Gamma density:

$$f(x) = \frac{x^{\gamma-1}e^{-x/\beta}}{\beta^{\gamma}\Gamma(\gamma)},$$

for $x > 0,\ \gamma > 0,\ \beta > 0$ and $f(x) = 0$ elsewhere. This includes the chisquare and exponential densities also.

$$\begin{aligned}\int_0^\infty [f(x)]^{2-\alpha}dx &= \frac{\int_0^\infty x^{(\gamma-1)(2-\alpha)}e^{-\frac{(2-\alpha)x}{\beta}}dx}{\beta^{(2-\alpha)\gamma}[\Gamma(\gamma)]^{2-\alpha}} \\ &= \frac{\beta^{\alpha-1}\Gamma[(\gamma-1)(2-\alpha)+1]}{[\Gamma(\gamma)]^{2-\alpha}}\end{aligned}$$

Hence

$$\begin{aligned}M_\alpha(f) &= \frac{1}{\alpha-1}\left[\frac{\beta^{\alpha-1}\Gamma[(\gamma-1)(2-\alpha)+1]}{[\Gamma(\gamma)]^{2-\alpha}} - 1\right] \\ &\text{and} \\ \lim_{\alpha\to 1} M_\alpha(f) &= \ln\beta + (1-\gamma)\psi(\gamma) - \ln\Gamma(\gamma) \\ &= \ln\beta \text{ for } \gamma = \alpha,\end{aligned}$$

where $\psi(\cdot)$ is a psi function or digamma function. It may be also noted that the continuous analogue of $M^*_{k,\alpha}(P)$ of (9) is

$$M^*_\alpha(f) = \frac{\ln(\int_{-\infty}^{\infty}[f(x)]^{2-\alpha}dx)}{\alpha-1},\ \alpha \neq 1,\ \alpha < 2.$$

For the gamma density $M^*_\alpha(f)$ also gives the same quantity as above, and $\ln\beta$ when $\alpha \to 1$.

It may be observed that the same procedure goes through even for the pathway model of Mathai (2005), in the scalar case, and $M_\alpha(f)$ as well as $M^*_\alpha(f)$ can be computed explicitly without much difficulty. The pathway model contains a large number of distributions of common use in various disciplines. But Shannon entropy S_k of (6) is quite difficult to evaluate, in the continuous analogues, even for many of the standard densities in common use due to the presence of $\ln f(x)$.